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Weinberg sum rules, four-quark condensates and chiral symmetry restoration

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Motivation

- QCD has non-trivial vacuum structure

$\rightsquigarrow$ can be characterized by condensates

- phase transitions (or rapid crossovers) at finite temperatures/densities

$\rightsquigarrow$ changes of condensates $\longrightarrow$ order parameters

- QCD sum rules connect hadronic properties (spectra)
  with quark-gluon properties (perturbation theory + condensates)

$\rightsquigarrow$ interest in four-quark condensates

Contents

- Generalized Weinberg sum rules and order parameters
- Evaluation of four-quark condensates (large N_c)
- Sizable temperatures and chemical potentials: resonance gas, phase transition

Weinberg sum rules

difference of vector ($\mathrm{Im}R^V$) and axial-vector ($\mathrm{Im}R^A$) spectra

(chiral partners, i.e. connected by chiral transformation)

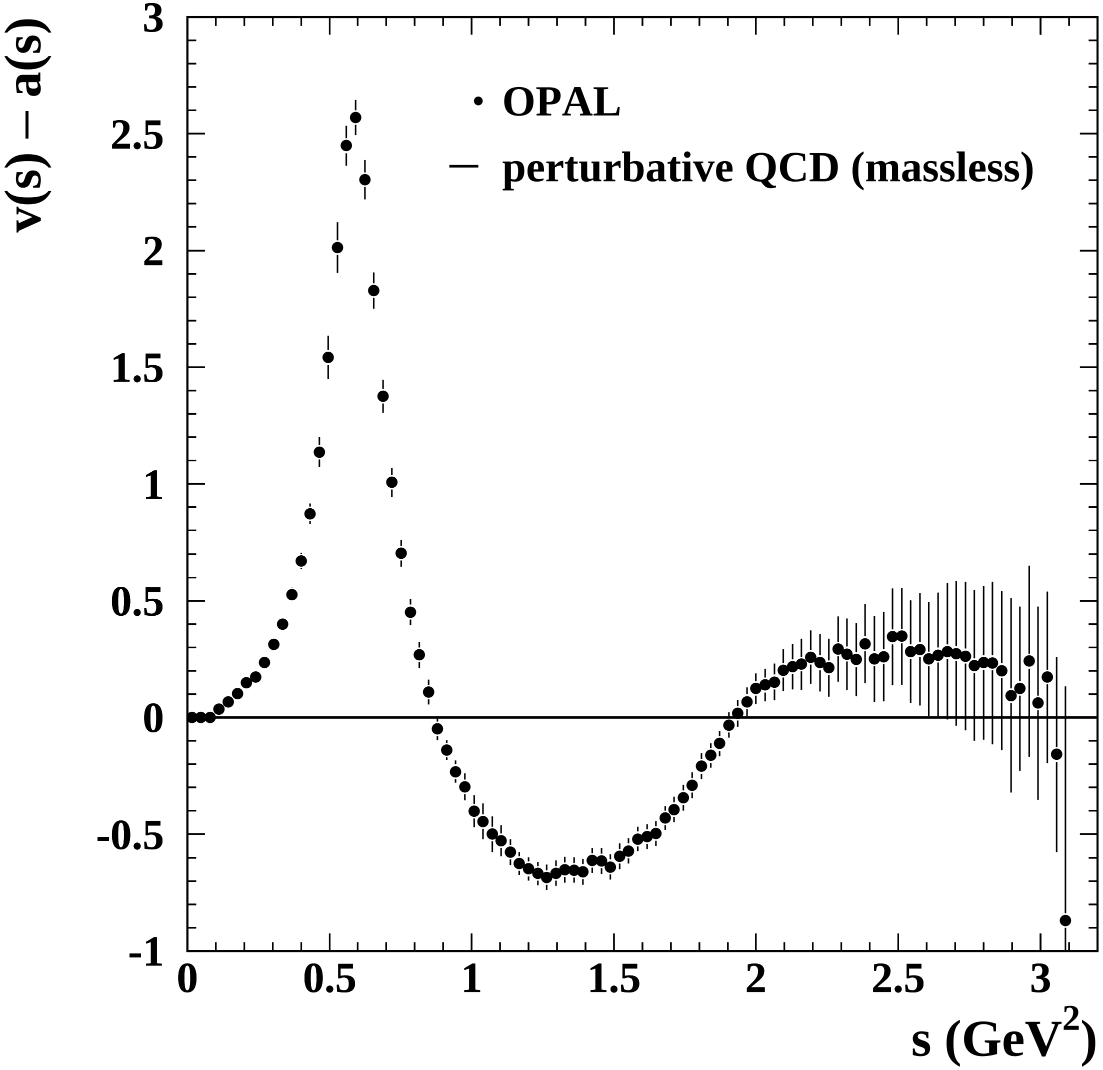
$$\frac{1}{\pi}\int_0^\infty ds\,\left(\mathrm{Im}R^V(s)-\mathrm{Im}R^A(s)\right) = F_\pi^2\,,$$

$$\frac{1}{\pi}\int_0^\infty ds\,s\,\left(\mathrm{Im}R^V(s)-\mathrm{Im}R^A(s)\right) = 0\,,$$

$$\frac{1}{\pi}\int_0^\infty ds\,s^2\,\left(\mathrm{Im}R^V(s)-\mathrm{Im}R^A(s)\right) = -\frac{1}{2}\,\pi\alpha_s\langle\mathcal{O}_{\chi\mathrm{SB}}\rangle\,,$$

with four-quark condensate

$$\langle\mathcal{O}_{\chi\mathrm{SB}}\rangle = \left\langle(\bar{u}\gamma_\mu\gamma_5\lambda^a u-\bar{d}\gamma_\mu\gamma_5\lambda^a d)^2-(\bar{u}\gamma_\mu\lambda^a u-\bar{d}\gamma_\mu\lambda^a d)^2\right\rangle$$



Eur. Phys. J.

C7 (1999) 571

- generalization to in-medium situations by Kapusta/Shuryak (1993)

$\hookrightarrow$ $\langle \mathcal{O}_{\chi \mathrm{SB}} \rangle$ is order parameter (somewhat oversimplified ...)

- closer connected to (in principle) measurable quantities (R^V and R^A) than two-quark condensate

$\hookrightarrow$ How to evaluate four-quark condensates in vacuum and medium?

- typical form: $\langle \bar{q} \Gamma \lambda_a q \, \bar{q} \Gamma' \lambda_a q \rangle$
  with spin-flavor matrices Γ and color Gell-Mann matrices λ_a

$\hookrightarrow$ Fierz transformations:
rewrite $\langle \bar{q} \Gamma \lambda_a q \, \bar{q} \Gamma' \lambda_a q \rangle$ as sum of $\langle \bar{q} \Gamma'' q \, \bar{q} \Gamma''' q \rangle$

- $\bar{q} \Gamma q$ connected to hadronic states

- vacuum: factorization assumption

$$\langle 0|\bar{q}\Gamma q\,\bar{q}\Gamma' q|0\rangle = \langle 0|\bar{q}\Gamma q|0\rangle\,\langle 0|\bar{q}\Gamma' q|0\rangle \qquad (-\text{ exchange terms})$$

— intuitive (mean-field approximation)

— justified in large-N_c limit (N_c: number of colors) [Novikov et al, 1984]

— in practice: matter of debate

- finite (low) temperature: medium described by pions, use current algebra

$\rightsquigarrow$ finite-temperature four-quark condensates do not factorize

[Eletsky 1993, Hatsuda et al 1993]

$\hookrightarrow$ other in-medium situations? (factorization would be useful)

strategy: study in-medium four-quark condensates at large N_c

$\longrightarrow$ understand non-factorization at finite (low) temperatures

$\longrightarrow$ figure out what happens for other in-medium situations common practice: evaluate in-medium condensates in linear-density approximation

$$\langle \mathcal{O} \rangle_{\rm med.} \approx \langle 0|\mathcal{O}|0\rangle + \sum_X \rho_X \langle X|\mathcal{O}|X\rangle$$

— density of medium-constituents ρ_X

— temperature: $X = \pi$, baryon density: $X = N$, more general: resonance gas

physical decomposition (and vacuum factorization):

$$\begin{aligned}
\langle \bar{q}\Gamma q\, \bar{q}\Gamma' q\rangle_{\rm med.} \approx\; & \langle 0|\bar{q}\Gamma q|0\rangle\, \langle 0|\bar{q}\Gamma' q|0\rangle \\
& + \rho_X \left(\langle X|\bar{q}\Gamma q|X\rangle\, \langle 0|\bar{q}\Gamma' q|0\rangle + \langle 0|\bar{q}\Gamma q|0\rangle\, \langle X|\bar{q}\Gamma' q|X\rangle \right) && \Big\{ \text{scatterer + spectator} \\
& + \rho_X \left(\langle X|\bar{q}\Gamma q|0\rangle\, \langle 0|\bar{q}\Gamma' q|X\rangle + \langle 0|\bar{q}\Gamma q|X\rangle\, \langle X|\bar{q}\Gamma' q|0\rangle \right) && \Big\{ \text{annihilation + creation} \\
& + \rho_X\, \langle X|\bar{q}\Gamma q\, \bar{q}\Gamma' q|X\rangle_{\rm connected} && \Big\{ \text{true scattering}
\end{aligned}$$

note: last two lines spoil factorization large-N_c evaluation ['t Hooft, Witten]: distinguish baryons (B) and mesons (M)

$$\langle \bar{q}\Gamma q\, \bar{q}\Gamma' q\rangle_{\mathrm{med.}} \approx \langle 0|\bar{q}\Gamma q|0\rangle\, \langle 0|\bar{q}\Gamma' q|0\rangle \qquad O(N_c^2)$$

$$+\rho_X\left(\langle X|\bar{q}\Gamma q|X\rangle\, \langle 0|\bar{q}\Gamma' q|0\rangle + \langle 0|\bar{q}\Gamma q|0\rangle\, \langle X|\bar{q}\Gamma' q|X\rangle\right) \begin{cases} O(N_c) \text{ for } X = M \\ O(N_c^2) \text{ for } X = B \end{cases}$$

$$+\rho_X\left(\langle X|\bar{q}\Gamma q|0\rangle\, \langle 0|\bar{q}\Gamma' q|X\rangle + \langle 0|\bar{q}\Gamma q|X\rangle\, \langle X|\bar{q}\Gamma' q|0\rangle\right) \begin{cases} O(N_c) \text{ for } X = M \\ \text{vanish for } X = B \end{cases}$$

$$+\rho_X\, \langle X|\bar{q}\Gamma q\, \bar{q}\Gamma' q|X\rangle_{\mathrm{connected}} \qquad \begin{cases} O(N_c^0) \text{ for } X = M \\ O(N_c) \text{ for } X = B \end{cases}$$

$\hookrightarrow$ no factorization at finite (low) temperatures = pion gas

(due to annihilation/creation terms)

$\hookrightarrow$ factorization justified in baryon dominated medium

(annihilation/creation terms vanish)

Sizable temperatures and chemical potentials:
resonance gas

$$\langle \mathcal{O}_{\chi\mathrm{SB}} \rangle_{\mathrm{med.}} \approx \langle 0 | \mathcal{O}_{\chi\mathrm{SB}} | 0 \rangle + \sum_{X = \mathrm{res.}} \rho_X \langle X | \mathcal{O}_{\chi\mathrm{SB}} | X \rangle$$

recall:

- meson contributions subleading in $1/N_c$

$$\langle M | \mathcal{O}_{\chi\mathrm{SB}} | M \rangle = o(N_c)$$

- baryon contributions in leading order, factorization possible

$$\langle B | \mathcal{O}_{\chi\mathrm{SB}} | B \rangle = O(N_c^2)$$

$$\langle \mathcal{O}_{\chi \mathrm{SB}} \rangle_{\mathrm{med.}} \approx 8 \langle \bar{q} q \rangle_{\mathrm{vac}}^2 \left(1 - \sum_{X=B} \frac{2 \sigma_X \rho_X}{F_\pi^2 M_\pi^2} \right)$$

- with sigma terms defined by

$$\sigma_X = 2 m_q \langle X | \bar{q} q | X \rangle$$

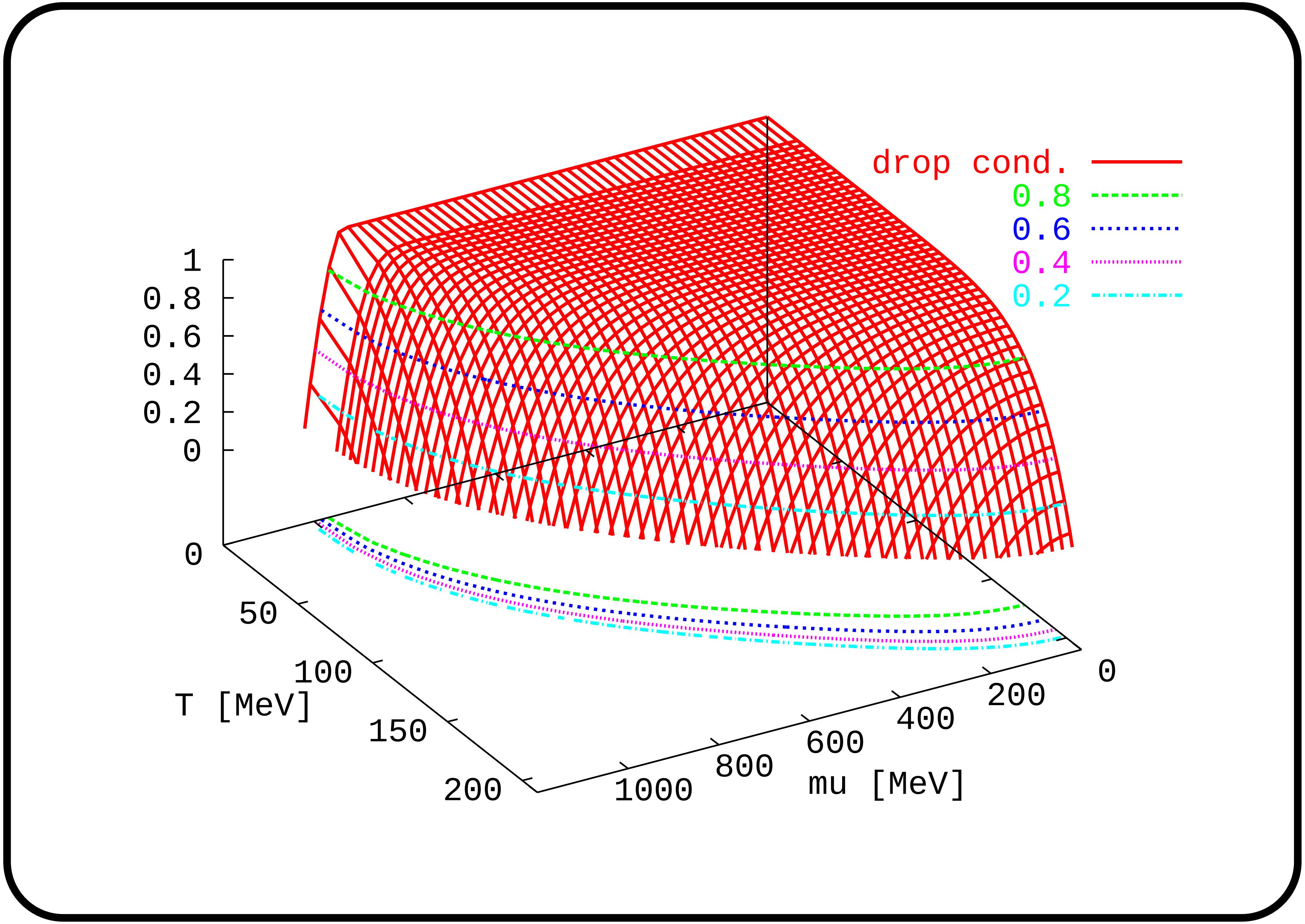
- estimate for sigma terms (Gerber/Leutwyler):

  non-relativistic quark model: $\bar{q} q \longrightarrow q^\dagger q$

$$\sigma_X \approx m_q \langle X | u^\dagger u + d^\dagger d | X \rangle = m_q \left(N_c - N_s \right)$$

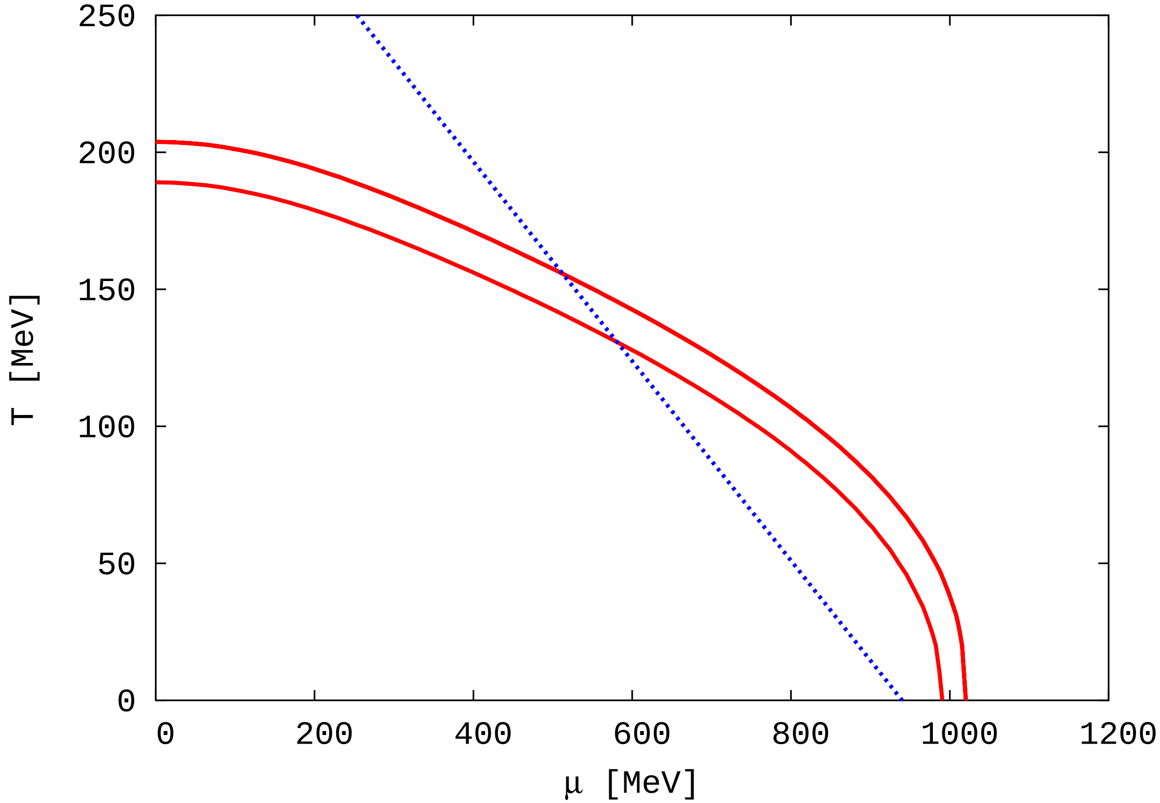
- underestimates σ_N by about factor of 2

$\rightsquigarrow$ uncertainty



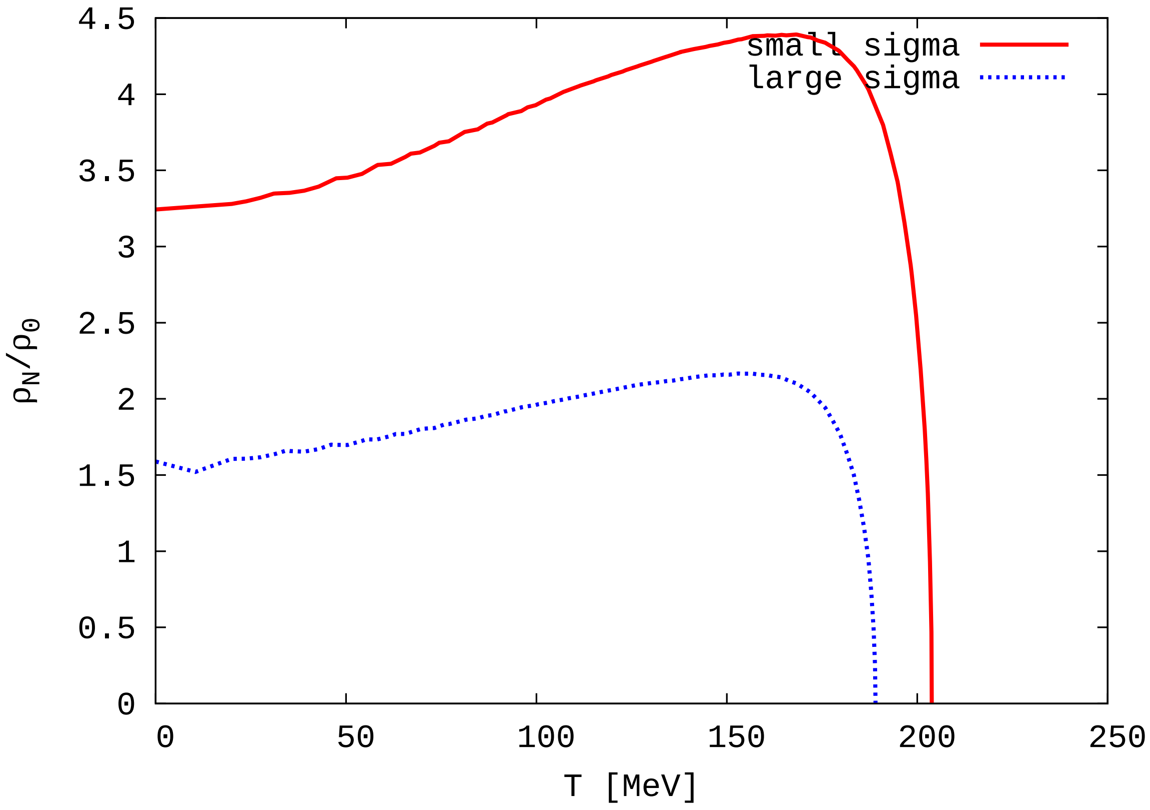
drop cond.
0.8
0.6
0.4
0.2
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
T [MeV]
1000
800
600
400
200
0
mu [MeV]

transition line



reasonable to the right of blue line (baryon dominated)

net density along transition line



Summary

- consistent picture for large number of colors

  (see also [S.L., Phys. Lett. B616, 203, 2005]):

  factorization assumption true for vacuum and baryon dominated systems,

  no factorization for finite temperatures

- important ingredient: physical interpretation of different terms

- uncertainty estimates $\rightsquigarrow$ O.K. (not shown here)

- resonance gas approximation for sizable temperature and chemical potential

$\hookrightarrow$ phase transition line

- especially relevant for CBM at GSI

additional slides

Fierz transformation for condensate which appears in generalized Weinberg SR

$$\langle(\bar{u}\gamma_\mu\gamma_5\lambda_a u - \bar{d}\gamma_\mu\gamma_5\lambda_a d)^2 - (\bar{u}\gamma_\mu\lambda_a u - \bar{d}\gamma_\mu\lambda_a d)^2\rangle =$$

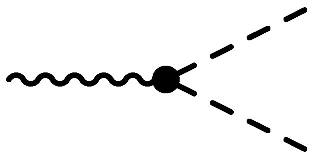
$$2\,\langle(\underbrace{\bar{u}u + \bar{d}d}_{\sim f_0\,(\sigma)})^2\rangle + 2\,\langle(\underbrace{\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d}_{\sim\pi^0})^2\rangle - 8\,\langle\underbrace{\bar{u}i\gamma_5 d}_{\sim\pi^-}\,\bar{d}i\gamma_5 u\rangle$$

$$+\,2\,\langle(\underbrace{\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d}_{\sim\eta,\eta'})^2\rangle + 2\,\langle(\underbrace{\bar{u}u - \bar{d}d}_{\sim\delta^0})^2\rangle - 8\,\langle\underbrace{\bar{u}d}_{\sim\delta^-}\,\bar{d}u\rangle$$

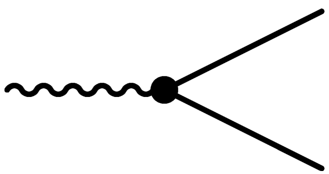
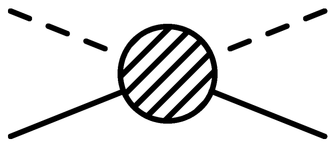
$$-\,\frac{2}{N_c}\,\langle(\underbrace{\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d}_{\sim\pi^0,a_1^0})^2 - (\underbrace{\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d}_{\sim\rho^0})^2\rangle$$

large-N_c scaling ['t Hooft 1974, Witten 1979]

- meson: $q\bar{q}$ state; mass $O(N_c^0)$ 
- baryon: N_c quarks; mass $O(N_c)$
- ratio of quark current $\bar{q}\Gamma q$ and corresponding hadronic field $O(N_c^{1/2})$
- mesonic interactions suppressed:

e.g.  $O(N_c^{-1/2})$

- meson-baryon interactions not suppressed:

 $O(N_c^{1/2})$;  $O(N_c^0)$

Uncertainty estimates

- pion gas: exact calculations possible using current algebra (and vacuum factorization) (Eletsky, Hatsuda/Koike/Lee (1992))

$$\langle \mathcal{O}_{\chi\mathrm{SB}} \rangle_{\text{pionic med.}} \quad = \quad \frac{8(N_c^2 - 1)}{N_c^2} \langle \bar{q} q \rangle_{\mathrm{vac}}^2 \left(1 - \frac{8 \rho_\pi}{3 F_\pi^2} \right)$$

$\hookrightarrow$ $\approx$ 10% correction

- nucleon gas: recall Fierz transformation

$\hookrightarrow$ corrections come from meson-nucleon scattering

$\hookrightarrow$ can estimate at least pion contribution and compare to factorized part (in-medium generalization of estimate of Shifman et al.)

Fierz transformation for condensate which appears in generalized Weinberg SR

$$\langle N|(\bar{u}\gamma_\mu\gamma_5\lambda_a u - \bar{d}\gamma_\mu\gamma_5\lambda_a d)^2 - (\bar{u}\gamma_\mu\lambda_a u - \bar{d}\gamma_\mu\lambda_a d)^2|N\rangle =$$

$$2\,\langle N|(\underbrace{\bar{u}u + \bar{d}d}_{\sim f_0\,(\sigma)})^2|N\rangle + 2\,\langle N|(\underbrace{\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d}_{\sim\pi^0})^2|N\rangle - 8\,\langle N|\underbrace{\bar{u}i\gamma_5 d}_{\sim\pi^-}\,\bar{d}i\gamma_5 u|N\rangle$$

$$+\,2\,\langle N|(\underbrace{\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d}_{\sim\eta,\eta'})^2|N\rangle + 2\,\langle N|(\underbrace{\bar{u}u - \bar{d}d}_{\sim\delta^0})^2|N\rangle - 8\,\langle N|\underbrace{\bar{u}d}_{\sim\delta^-}\,\bar{d}u|N\rangle$$

$$-\,\frac{2}{N_c}\,\langle N|(\underbrace{\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d}_{\sim\pi^0,a_1^0})^2 - (\underbrace{\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d}_{\sim\rho^0})^2|N\rangle$$

$$\langle N|(\bar{u}\gamma_\mu\gamma_5\lambda_a u - \bar{d}\gamma_\mu\gamma_5\lambda_a d)^2 - (\bar{u}\gamma_\mu\lambda_a u - \bar{d}\gamma_\mu\lambda_a d)^2|N\rangle =$$

$$4\,\langle N|\bar{u}u + \bar{d}d|N\rangle\,\langle \bar{u}u + \bar{d}d\rangle_{\mathrm{vac}}$$

$$+\,2\,\langle N|(\underbrace{\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d}_{\sim\pi^0})^2|N\rangle - 8\,\langle N|\underbrace{\bar{u}i\gamma_5 d}_{\sim\pi^-}\,\bar{d}i\gamma_5 u|N\rangle$$

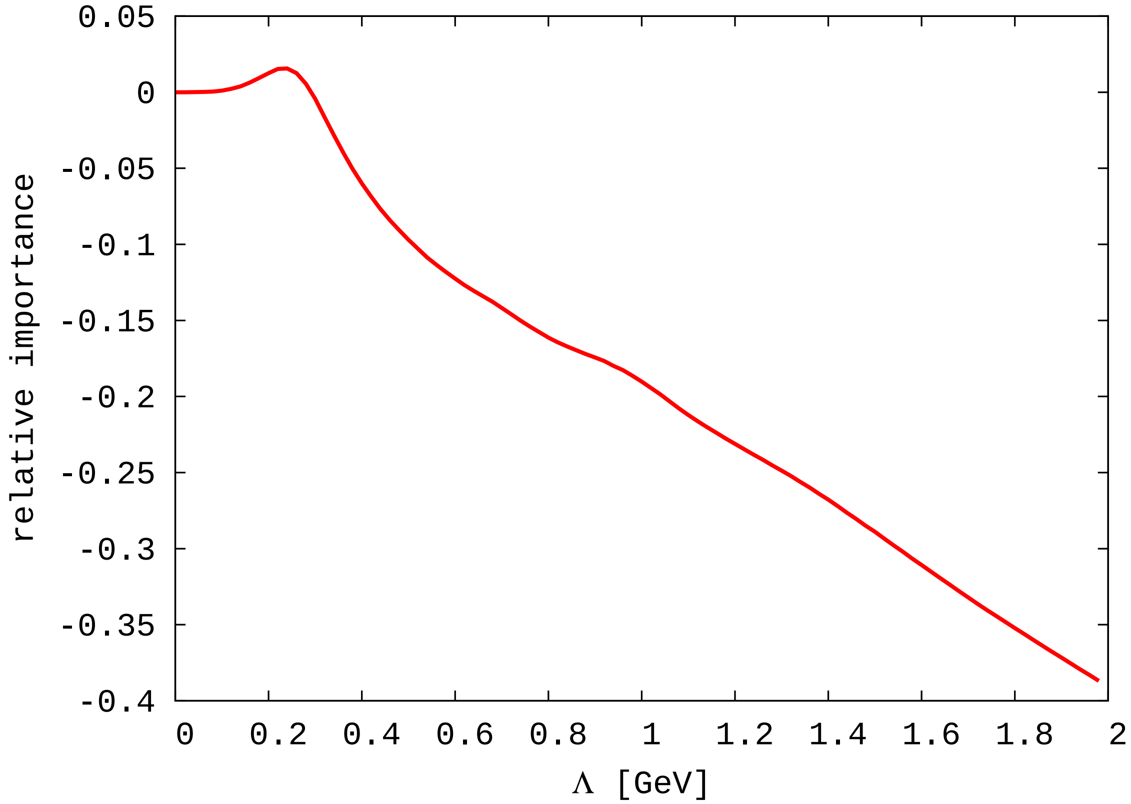
$+$ contributions from other mesons

$\hookrightarrow$ estimate correction to large-N_c result by pionic contribution:

$$\frac{2\,\langle N|(\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d)^2|N\rangle - 8\,\langle N|\bar{u}i\gamma_5 d\,\bar{d}i\gamma_5 u|N\rangle}{4\,\langle N|\bar{u}u + \bar{d}d|N\rangle\,\langle \bar{u}u + \bar{d}d\rangle_{\mathrm{vac}}}$$

$\hookrightarrow$ pion-nucleon scattering integrated over all momenta (condensate is local)

$\hookrightarrow$ depends on cutoff/separation between non-perturbative and perturbative regime



0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
relative importance
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Λ [GeV]

Uncertainty estimates

- pion gas: exact calculations possible using current algebra (and vacuum factorization) (Eletsky, Hatsuda/Koike/Lee (1992))

$$\langle \mathcal{O}_{\chi \mathrm{SB}} \rangle_{\text{pionic med.}} = \frac{8(N_c^2 - 1)}{N_c^2} \langle \bar{q} q \rangle_{\text{vac}}^2 \left(1 - \frac{8 \rho_\pi}{3 F_\pi^2} \right)$$

$\hookrightarrow$ $\approx$ 10% correction

- nucleon gas: recall Fierz transformation

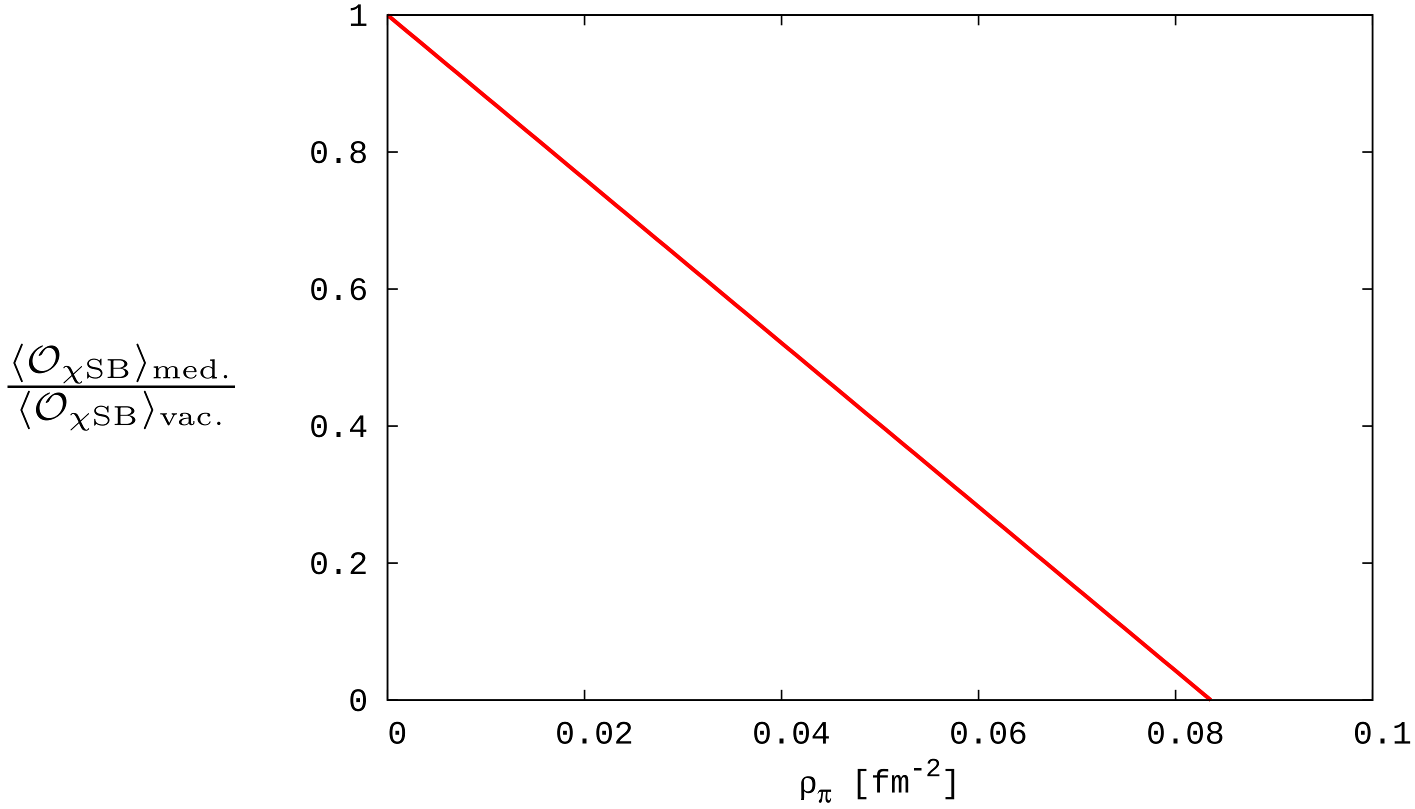
$\hookrightarrow$ corrections come from meson-nucleon scattering

$\hookrightarrow$ can estimate at least pion contribution and compare to factorized part (in-medium generalization of estimate of Shifman et al.)

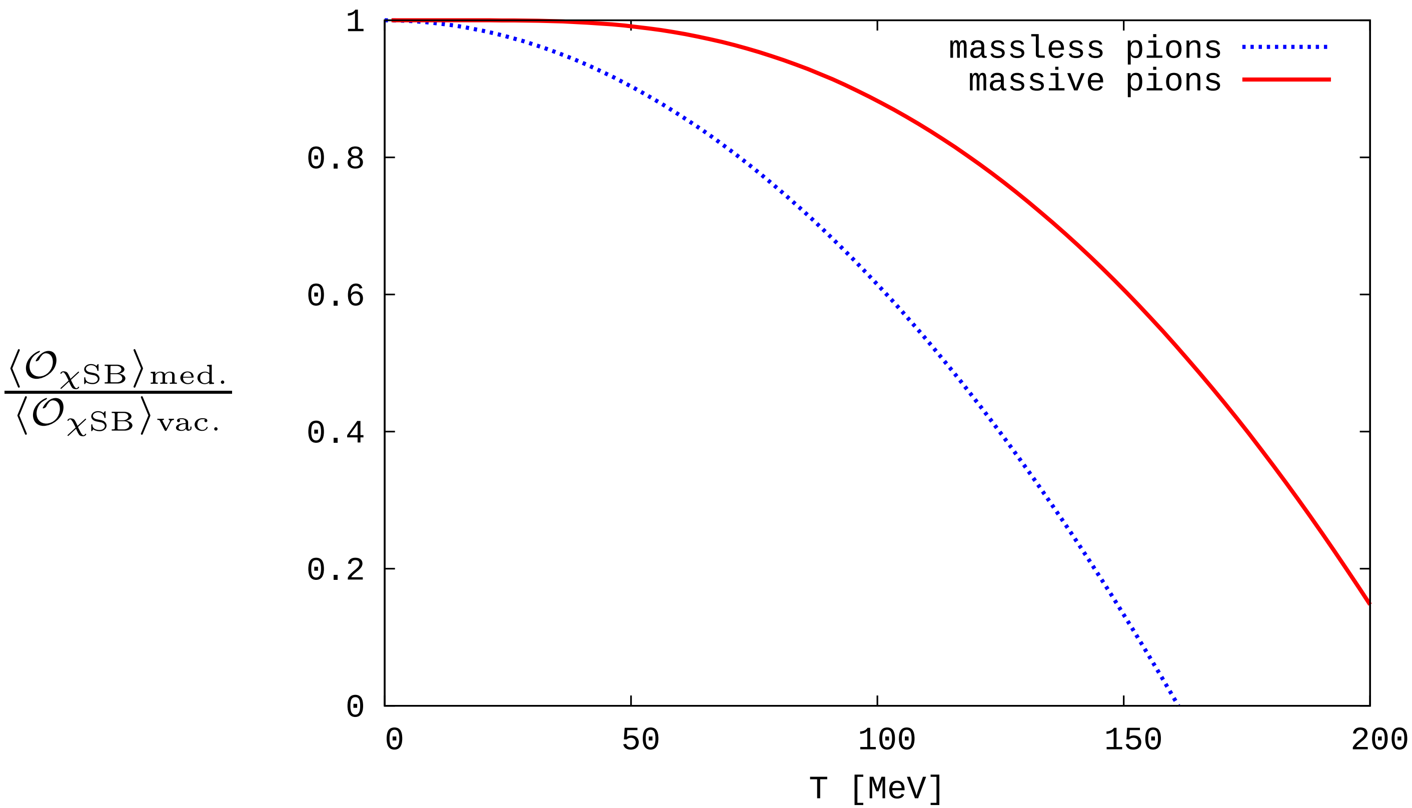
$\hookrightarrow$ at most 20% correction

Low temperatures: pion gas

drop of condensate as function of pion density



drop of condensate as function of temperature



Finite baryo-chemical potential: nucleon gas

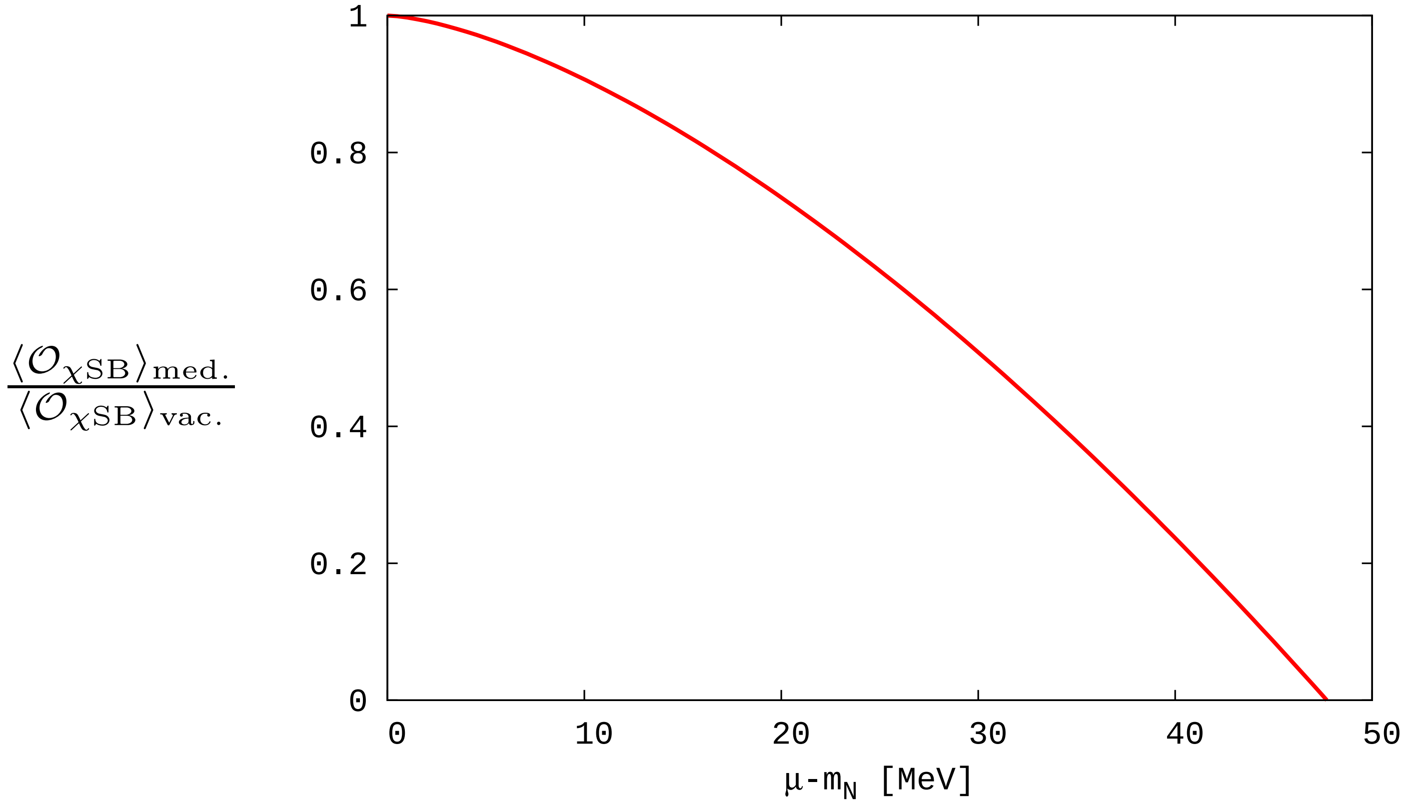
recall: here factorization possible at large N_c

$$\langle \mathcal{O}_{\chi\mathrm{SB}} \rangle_{\text{nuclear med.}} = 8 \langle \bar{q} q \rangle^2_{\text{vac}} \left(1 - \frac{2 \sigma_N \rho_N}{F_\pi^2 M_\pi^2} \right)$$

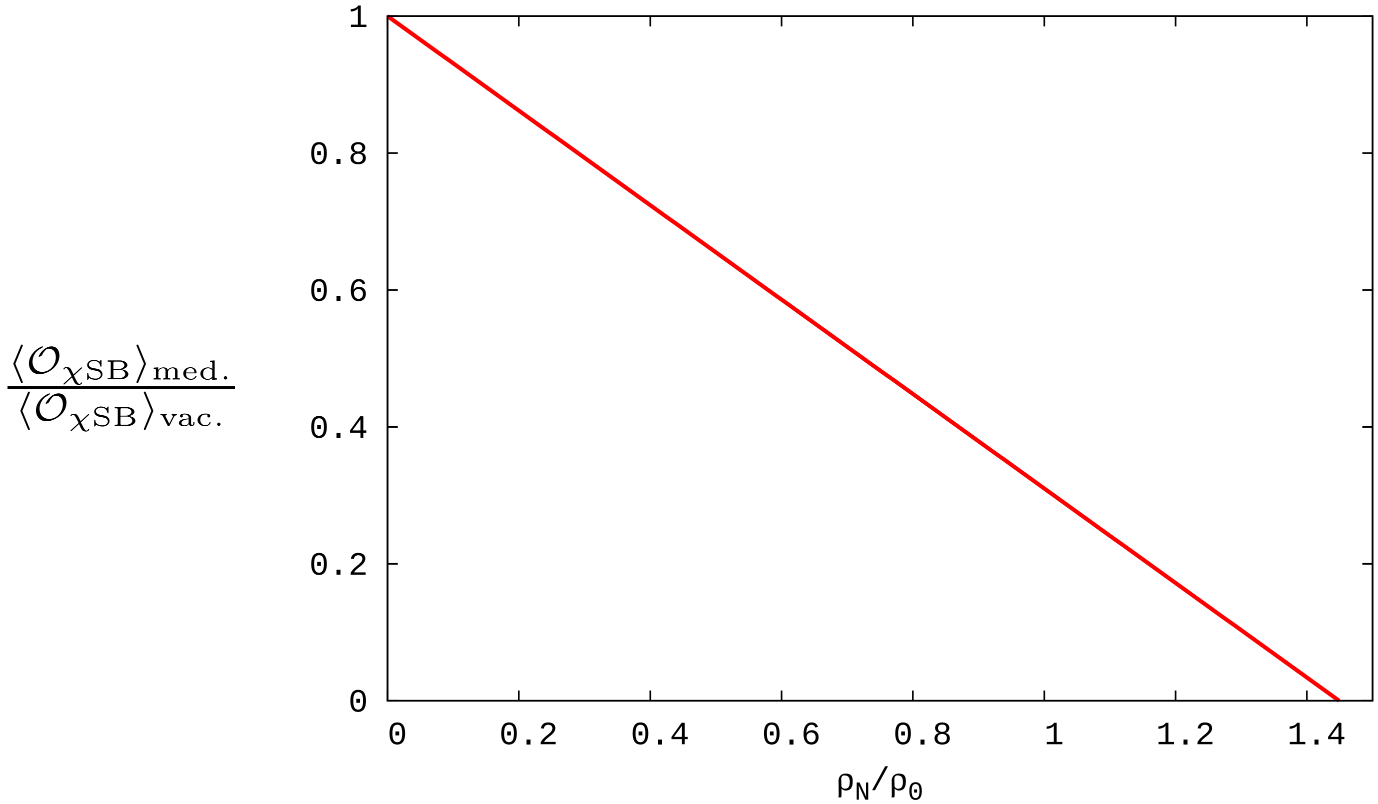
with nucleon sigma term $\sigma_N \approx$ 45 MeV defined by

$$\sigma_N = 2 m_q \langle N | \bar{q} q | N \rangle$$

drop of condensate as function of baryo-chemical potential



drop of condensate as function of nucleon density



presumably effects beyond linear-density approximation become important

at least above ρ_0

Appropriate for which chemical potentials/temperatures?

$$\rho_B \sim e^{-(M_B-\mu)/T}\,, \qquad \rho_M \sim e^{-M_M/T}$$

- $M_B = O(N_c)$, $M_M = O(1)$

1. $\mu, T = O(1)$

$\hookrightarrow$ $\rho_B \sim e^{-N_c}$ ($\ll 1/N_c$ for large N_c)

$\hookrightarrow$ baryonic contribution even suppressed!

2. $\mu \sim O(N_c)$, $T = O(1)$

$\hookrightarrow$ $\rho_B = O(1)$, no suppression

- also without N_c-counting:

  for $\mu > M_N - T$ enhancement of baryons relative to (heavy) mesons

$\hookrightarrow$ appropriate for large chemical potentials and temperatures

(for T too small: no resonances, only nucleons) exchange terms are always subleading in $1/N_c$:

$$\bar{q}_i \Gamma q_i \; \bar{q}_j \Gamma' q_j \longrightarrow \delta_{ii}\delta_{jj} = N_c^2$$

whereas

$$\bar{q}_i \Gamma q_i \; \bar{q}_j \Gamma' q_j \longrightarrow \delta_{ij}\delta_{ij} = N_c$$

Four-quark condensates in vacuum:

Determination from QCD sum rules possible?

Finite energy sum rules for ρ-meson (somewhat oversimplified ...):

$$\frac{1}{\pi}\int_0^{s_0} ds\, \mathrm{Im} R^{\mathrm{RES}}(s) = \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right) s_0\,,$$

$$\frac{1}{\pi}\int_0^{s_0} ds\, s\, \mathrm{Im} R^{\mathrm{RES}}(s) = \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right) \frac{s_0^2}{2} - \frac{1}{24}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle\,,$$

$$\frac{1}{\pi}\int_0^{s_0} ds\, s^2\, \mathrm{Im} R^{\mathrm{RES}}(s) = \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right) \frac{s_0^3}{3} - \frac{112}{81}\pi\bar{\alpha}_s \frac{3}{N_c}\langle \mathcal{O}_{4-\mathrm{q.}}\rangle$$

sum rules have intrinsic uncertainties due to simplifications

$\hookrightarrow$ insert reasonable values and attribute 5% uncertainty (for mass, ...)

$$\frac{1}{\pi}\int_0^{s_0} ds\, \mathrm{Im} R^{\mathrm{RES}}(s) = \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right) s_0$$

$$17 \pm 1.7 \approx 18 \pm 1.8$$

$$\frac{1}{\pi}\int_0^{s_0} ds\, s\, \mathrm{Im} R^{\mathrm{RES}}(s) = \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right) \frac{s_0^2}{2} - \frac{1}{24}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle$$

$$1. \pm 0.2 \approx 1.2 \pm 0.2 - 0.05$$

$$\frac{1}{\pi}\int_0^{s_0} ds\, s^2\, \mathrm{Im} R^{\mathrm{RES}}(s) = \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right) \frac{s_0^3}{3} - \frac{112}{81}\pi\bar{\alpha}_s \frac{3}{N_c}\langle \mathcal{O}_{4-\mathrm{q.}} \rangle$$

$$6. \pm 2. \approx 10. \pm 3. - 0.3$$

standard/easy: determination of hadronic parameters from sum rules

complicated: determination of condensates from physical spectrum